Percolation for the Vacant Set of Random Interlacements

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Abstract

We investigate random interlacements on $\mathbb{Z}^d$, $d \geq 3$. This model, recently introduced in [8], corresponds to a Poisson cloud on the space of doubly infinite trajectories modulo time shift tending to infinity at positive and negative infinite times. A nonnegative parameter u measures how many trajectories enter the picture. Our main interest lies in the percolative properties of the vacant set left by random interlacements at level u. We show that for all $d \geq 3$ the vacant set at level u percolates when u is small. This solves an open problem of [8], where this fact has only been established when $d \geq 7$. It also completes the proof of the nondegeneracy in all dimensions $d \geq 3$ of the critical parameter u_* of [8].

Introduction

The present work investigates the model of random interlacements on $\mathbb{Z}^d$, $d \geq 3$, introduced in [8]. Informally this is a translation-invariant model that describes the microscopic structure left in the bulk by a random walk on a large discrete torus or on a discrete cylinder with base a large discrete torus when the walk is run for a time proportional to the number of sites in the torus or to the square of the number of sites in the base (cf. [7, 11]. The main purpose of this article is to answer an open question of [8] and show that for small $u > 0$ the vacant set at level u in $\mathbb{Z}^d$ left by random interlacements does percolate. In [8] this had only been proved to be the case when $d \geq 7$. The present work shows that small dimensions behave in a similar fashion.

We now describe the model somewhat informally and refer to Section 1 for precise definitions. Random interlacements consist of a cloud of paths constituting a Poisson point process on the space of doubly infinite $\mathbb{Z}^d$-valued trajectories modulo time shift tending to infinity at positive and negative infinite times. A nonnegative parameter u plays in essence the role of a multiplicative factor of the intensity measure of this Poisson point process. In a standard fashion one constructs on the same space $(\Omega, \mathcal{A}, P)$ (see (1.9) below), the whole family $\mathcal{I}^u$, $u \geq 0$, of random interlacements at level $u \geq 0$ (cf. (1.16)). They are the traces on $\mathbb{Z}^d$ of the cloud

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of trajectories modulo time shift "up to level u." The random subsets $\mathcal{I}^u$ increase with u and for $u > 0$ are infinite random connected subsets of $\mathbb{Z}^d$, ergodic under space translations (cf. [8, theorem 2.1]. The complement of $\mathcal{I}^u$ in $\mathbb{Z}^d$ is denoted by $\mathcal{V}^u$. It is the so-called vacant set at level u (cf. (1.17)). As shown in (2.16) of [8], the law Q_u on $\{0, 1\}^{\mathbb{Z}^d}$ of the indicator function of $\mathcal{V}^u$ is characterized by the property

$$(0.1)\quad Q_u[Y_x = 1 \text{ for all } x \in K] = \exp\{-u \operatorname{cap}(K)\} \text{ for all finite sets } K \subseteq \mathbb{Z}^d,$$

where Y_x, $x \in \mathbb{Z}^d$, stand for the canonical coordinates on $\{0, 1\}^{\mathbb{Z}^d}$ and $\operatorname{cap}(K)$ for the capacity of K (cf. (1.5)).

Our main focus here lies in the percolative properties of $\mathcal{V}^u$. For this purpose it is convenient to consider the nonincreasing function

$$(0.2)\quad \eta(u) = \mathbb{P}[0 \text{ belongs to an infinite connected component of } \mathcal{V}^u], \quad u \geq 0.$$

With corollary 2.3 of [8], one knows that the $\mathbb{P}$-almost sure presence and absence of an infinite connected component (i.e.,"infinite cluster") in $\mathcal{V}^u$ are equivalent to $\eta(u) > 0$ and $\eta(u) = 0$, respectively. One then introduces the critical parameter

$$(0.3)\qquad u_* = \inf\{u \geq 0 : \eta(u) = 0\} \in [0, \infty].$$

The main results of [8] (cf. theorems 3.5 and 4.3) show that

$$(0.4)\qquad u_* < \infty \text{ for } d \geq 3 \quad \text{and} \quad u_* > 0 \text{ for } d \geq 7;$$

i.e., $\mathcal{V}^u$ does not percolate for large u, and at least when $d \geq 7$, percolates for small u. The main result in the present article (cf. Theorem 3.4) shows that

$$(0.5)\qquad u_* > 0 \quad \text{for } d \geq 3,$$

and even that $\mathcal{V}^u$ percolates in planes for small u. This solves an open problem of [8] and proves that u_* is nondegenerate in all dimensions. Let us also mention that with theorem 1.1 and corollary 1.2 of [10], it is known when $\eta(u) > 0$, i.e., when $\mathcal{V}^u$ percolates, the infinite cluster is almost surely unique and that η is continuous on $[0, u_*]$ and has at most one point of discontinuity at u_*. It is at present unknown whether $\mathcal{V}^{u_*}$ percolates or not.

Let us give some comments on the proof of (0.5). The difficulty in proving (0.5) stems from the fact that the usual Peierls-type arguments that require a good enough exponential bound on $\mathbb{P}[\mathcal{I}^u \supseteq A]$ in terms of the cardinality $|A|$ for A finite in $\mathbb{Z}^2$ (viewed as a subset of $\mathbb{Z}^d$) so far only work when $d \geq 18$; see [8, remark 2.5(3)]. In fact, when $d = 3$, there is no such exponential bound; cf. (1.20). This difficulty is closely related to the long-range dependence present in the model: as shown in (1.68) of [8], the correlation of the events $\{x \in \mathcal{V}^u\}$ and $\{y \in \mathcal{V}^u\}$ decays as $c(u)|x-y|^{-(d-2)}$, when $|x-y|$ tends to infinity. To bypass this obstruction we employ a renormalization technique that is different but has a similar flavor to the methods of [8, sec. 3]. To prove (0.5) we show (cf. (3.25) and (3.26)) that for small u the probability that a $*$-circuit of $\mathcal{I}^u \cap \mathbb{Z}^2$ surrounds the origin is smaller than 1 (we refer to the beginning of Section 1 for the definition of $*$-paths). For

this purpose we develop estimates showing that for small u the presence of long $*$-paths in $\mathcal{I}^u \cap \mathbb{Z}^2$ is unlikely. We consider a sequence of functions

$$q_n(u) \text{ “=” } \mathbb{P}\text{-probability that } \mathcal{I}^u \cap \mathbb{Z}^2 \text{ contains a } *\text{-path from a given block of size } L_n \text{ to the complement of its } L_n\text{-neighborhood for } n \geq 0 \tag{0.6}$$

(we refer to (2.7) and (2.8) for the precise expression), with the aim of proving that for small u, $q_n(u)$ decays with n at least as an inverse power of L_n. The sequence of length scales L_n, $n \geq 0$, in (0.6) grows rapidly, and (see (2.1), (2.2)):

$$L_n \approx L_0^{(1+a)^n}, \quad n \geq 0 \text{ with } a = \tfrac{1}{100}. \tag{0.7}$$

We derive a recurrence relation bounding $q_{n+1}(u_{n+1})$ in terms of $q_n(u_n)$ along a decreasing sequence u_n such that (cf. (2.67))

$$u_{n+1} = \left(1 + \frac{1}{\log L_n}\right)^{-1} u_n \quad \text{for } n \geq 0. \tag{0.8}$$

As a result of (0.7) this sequence converges to a positive value $u_\infty > 0$. The recurrence relation is based on Proposition 2.1 and hinges on the “sprinkling technique” of [8], where more independent paths are thrown in, with the purpose of dominating long-range interactions present in the model. In the proof of theorem 4.3 in [8], when showing $u_* > 0$ for $d \geq 7$, these long-range interactions could be bounded in a rather primitive way, with not too dire consequences thanks to the assumption $d \geq 7$. An important contribution of the present work is that we are able to control these interactions even in the case of small dimensions; see also Remark 2.3(2). The result of the renormalization scheme (cf. Theorem 2.5) is that for suitable dimension-dependent constants c, c', and c'', if we can find

$$L_0 \geq c \quad \text{and} \quad u_0 \geq c' \frac{(\log L_0)^2}{L_0^{d-2}} \quad \text{such that} \quad q_0(u_0) \leq c'' L_0^{-(1+2a)}, \tag{0.9}$$

then

$$\text{for all } n \geq 0, \quad q_n(u_n) \leq c'' L_n^{-(1+2a)}. \tag{0.10}$$

This procedure essentially reduces the proof of (0.5) to checking (0.9). This step is carried out in Theorem 3.1, where it is shown that

$$\lim_{L_0 \to \infty} L_0^\rho q_0(u_0) = 0 \quad \text{for all } \rho > 0 \text{ with } u_0 = \frac{c'(\log L_0)^2}{L_0^{d-2}}. \tag{0.11}$$

The two-dimensional character of the event in the right-hand side of (0.6) plays a crucial role here. Replacing $\mathbb{Z}^2$ with $\mathbb{Z}^d$ would still lead to a rather similar recurrence relation between $q_{n+1}(u_{n+1})$ and $q_n(u_n)$. However, one could not initiate the induction in this modified setup (cf. Remark 2.6; (0.11) would be replaced with $\lim_{L_0 \to \infty} q_0(u_0) = 1$). Interestingly, the proof of (0.11) relies on arguments reminiscent of some of the steps that appear in the derivation of lower bounds on the disconnection times of discrete cylinders by random walks; see section 2 of [1] or section 5 of [9].

We will now describe the organization of this paper.

In Section 1 we introduce some notation and recall useful facts concerning random interlacements.

In Section 2 we develop the renormalization scheme. The induction step is carried out in Proposition 2.1. The application of the induction step to the proof of the fact that (0.10) is a consequence of (0.9) appears in Proposition 2.4 and Theorem 2.5.

In Section 3 we prove (0.11) in Theorem 3.1. This enables us to initiate the induction and yields (0.10) for a decreasing sequence with positive limit u_∞. As a consequence, we show in Theorem 3.4 that for small $u > 0$, $\mathbb{P}$-almost surely $\mathcal{V}^u \cap \mathbb{Z}^2$ percolates, which in particular yields (0.5).

Finally, let us explain the convention we use for constants. Throughout the text, c and c' denote positive constants that depend solely on d, with values changing from place to place. The numbered constants $c_0, c_1, \ldots$ are fixed and refer to the value at their first appearance in the text. Dependence of constants on additional parameters appears in the notation.

1 Notation and Some Facts about Random Interlacements

The main purpose of this section is to introduce additional notation and recall various useful facts concerning random interlacements.

We let $|\cdot|$ and $|\cdot|_\infty$, respectively, stand for the Euclidean and the ℓ^∞-distance on $\mathbb{Z}^d$. Unless explicitly mentioned we assume $d \geq 3$ throughout this paper. We say that x, y in $\mathbb{Z}^d$ are neighbors, respectively $*$-neighbors, if $|x-y| = 1$, respectively $|x-y|_\infty = 1$. By finite path, respectively finite $*$-path, we mean a sequence $x_0, x_1, \ldots, x_N$ in $\mathbb{Z}^d$, $N \geq 0$, such that x_i and x_{i+1} are neighbors, respectively $*$-neighbors, for each $0 \leq i < N$. We also sometimes write path, or $*$-path, in place of finite path, or finite $*$-path, when this causes no confusion. With $B(x, r)$ and $S(x, r)$ we denote the closed $|\cdot|_\infty$-ball and $|\cdot|_\infty$-sphere with radius $r \geq 0$ and center $x \in \mathbb{Z}^d$. For A, B subsets of $\mathbb{Z}^d$, we write $A + B$ for the set of elements $x + y$ with x in A and y in B, and $d(A, B) = \inf\{|x-y|_\infty : x \in A,\ y \in B\}$ for the mutual ℓ^∞-distance between A and B. When A is a singleton $\{x\}$, we write $d(x, B)$ for simplicity. The notation $U \Subset \mathbb{Z}^d$ means that U is a finite subset of $\mathbb{Z}^d$. Given U a subset of $\mathbb{Z}^d$, we denote by $|U|$ the cardinality of U, with ∂U the boundary of U and $\partial_{\text{int}} U$ the interior boundary of U:

$$\begin{aligned} \partial U &= \{x \in U^c : \exists y \in U,\ |x-y| = 1\}, \\ \partial_{\text{int}} U &= \{x \in U : \exists y \in U^c,\ |x-y| = 1\}. \end{aligned} \tag{1.1}$$

The canonical basis of $\mathbb{R}^d$ is denoted by $(e_i)_{1 \leq i \leq d}$, and we tacitly identify $\mathbb{Z}^2$ with $\mathbb{Z}e_1 + \mathbb{Z}e_2 \subseteq \mathbb{Z}^d$.

We write W_+ for the space of nearest-neighbor $\mathbb{Z}^d$-valued trajectories defined for nonnegative times and tending to infinity. We denote by $\mathcal{W}_+$, X_n, $n \geq 0$, and $\mathcal{F}_n$, $n \geq 0$, the canonical σ-algebra, the canonical process, and canonical filtration

on W_+. We let θ_n, $n \geq 0$, stand for the canonical shift on W_+. Since $d \geq 3$, a simple random walk on $\mathbb{Z}^d$ is transient and for $x \in \mathbb{Z}^d$ we denote with P_x the restriction of the canonical law of a simple random walk starting at x to the set W_+, which has full measure. We write E_x for the corresponding expectation. We also define $P_\rho = \sum_{x \in \mathbb{Z}^d} \rho(x) P_x$ when ρ is a measure on $\mathbb{Z}^d$, and write E_ρ for the corresponding expectation. Given $U \subseteq \mathbb{Z}^d$, we let H_U, $\tilde{H}_U$, and T_U stand for the respective entrance time, hitting time of U, and exit time from U:

$$H_U = \inf\{n \geq 0 : X_n \in U\}, \quad \tilde{H}_U = \inf\{n \geq 1 : X_n \in U\}, \tag{1.2}$$
$$T_U = \inf\{n \geq 0 : X_n \notin U\}.$$

In case of a singleton $U = \{x\}$, we write H_x or $\tilde{H}_x$ for simplicity.

We denote with $g(\cdot, \cdot)$ the Green function of the walk

$$g(x, y) = \sum_{n \geq 0} P_x[X_n = y], \quad x, y \in \mathbb{Z}^d, \tag{1.3}$$

which is symmetric in its two variables, and $g(y) = g(0, y)$ so that $g(x, y) = g(y - x)$, due to translation invariance. Given $K \Subset \mathbb{Z}^d$ we write e_K for the equilibrium measure of K and $\mathrm{cap}(K)$ for the capacity of K, so that

$$e_K(x) = \begin{cases} P_x[\tilde{H}_K = \infty] & \text{for } x \in K \\ 0 & \text{for } x \notin K, \end{cases} \tag{1.4}$$

$$\mathrm{cap}(K) = e_K(\mathbb{Z}^d) = \sum_{x \in K} P_x[\tilde{H}_K = \infty]. \tag{1.5}$$

It is straightforward to see from (1.5) that the capacity is subadditive in the sense that $\mathrm{cap}(K \cup K') \leq \mathrm{cap}(K) + \mathrm{cap}(K')$ for K, K' finite subsets of $\mathbb{Z}^d$. Further, the probability of entering K can be expressed as

$$P_x[H_K < \infty] = \sum_{y \in K} g(x, y)\, e_K(y) \quad \text{for } x \in \mathbb{Z}^d. \tag{1.6}$$

One also has the bounds (see, e.g., (1.9) of [8]):

$$\frac{\sum_{y \in K} g(x, y)}{\sup_{z \in K}(\sum_{y \in K} g(z, y))} \leq P_x[H_K < \infty] \leq \frac{\sum_{y \in K} g(x, y)}{\inf_{z \in K}(\sum_{y \in K} g(z, y))} \tag{1.7}$$

for x in $\mathbb{Z}^d$, from which we can infer with the help of classical bounds on the Green function (cf. [5, p. 31]) that

$$c L^{d-2} \leq \mathrm{cap}(B(0, L)) \leq c' L^{d-2} \quad \text{for } L \geq 1. \tag{1.8}$$

To introduce random interlacements, we need some further objects. We denote by W the space of doubly infinite nearest-neighbor $\mathbb{Z}^d$-valued trajectories, which tend to infinity at positive and negative infinite times, and with W^* the space of equivalence classes of trajectories in W modulo time shift. The canonical

projection from W onto W^* is denoted by π^*. We endow W with its canonical σ-algebra generated by the canonical coordinates X_n, $n \in \mathbb{Z}$, and W^* with $\mathcal{W}^* = \{A \subseteq W^* : (\pi^*)^{-1}(A) \in \mathcal{W}\}$, the largest σ-algebra on W^* for which $\pi^* : (W, \mathcal{W}) \to (W^*, \mathcal{W}^*)$ is measurable.

We will now describe the space $(\Omega, \mathcal{A}, \mathbb{P})$ where random interlacements are defined. We consider the space of point measures on $W^* \times \mathbb{R}_+$:

$$\Omega = \Big\{\omega = \sum_{i \geq 0} \delta_{(w_i^*, u_i)} \text{ with } (w_i^*, u_i) \in W^* \times \mathbb{R}_+ \text{ for } i \geq 0 \text{ and } \omega(W_K^* \times [0,u]) < \infty \text{ for any } K \Subset \mathbb{Z}^d,\ u \geq 0\Big\}, \tag{1.9}$$

where for $K \Subset \mathbb{Z}^d$, $W_K^* \subseteq W^*$ is the set of trajectories modulo time shift that enter K:

$$W_K^* = \pi^*(W_K) \quad \text{and} \quad W_K = \{w \in W : \text{for some } n \in \mathbb{Z},\ X_n(w) \in K\}. \tag{1.10}$$

We endow Ω with the σ-algebra $\mathcal{A}$ generated by the evaluation maps $\omega \to \omega(D)$, where D runs over the product σ-algebra $\mathcal{W}^* \times \mathcal{B}(\mathbb{R}_+)$. We denote with $\mathbb{P}$ the probability on $(\Omega, \mathcal{A})$, which is the Poisson point measure with intensity $\nu(dw^*)du$, giving finite mass to the sets $W_K^* \times [0,u]$, for $K \Subset \mathbb{Z}^d$, $u \geq 0$, where ν is the unique σ-finite measure on $(W^*, \mathcal{W}^*)$ such that for any $K \Subset \mathbb{Z}^d$ (cf. [8, theorem 1.1]:

$$\mathbb{1}_{W_K^*} \nu = \pi^* \circ Q_K, \tag{1.11}$$

with Q_K the finite measure on W_K^0, the subset of W_K of trajectories that enter K for the first time at time 0 such that for $A, B \in \mathcal{W}_+$, $x \in \mathbb{Z}^d$ (see (1.4) for the notation):

$$Q_K[(X_{-n})_{n \geq 0} \in A,\ X_0 = x,\ (X_n)_{n \geq 0} \in B] = P_x[A \mid \tilde{H}_K = \infty] e_K(x) P_x[B]. \tag{1.12}$$

Given $K \Subset \mathbb{Z}^d$, $u \geq 0$, one further defines on $(\Omega, \mathcal{A})$ the random point process with values in the set of finite point measures on $(W_+, \mathcal{W}_+)$:

$$\mu_{K,u}(\omega) = \sum_{i \geq 0} \delta_{(w_i^*)^{K,+}} \mathbb{1}_{\{w_i^* \in W_K^*, u_i \leq u\}} \quad \text{for } \omega = \sum_{i \geq 0} \delta_{(w_i^*, u_i)}, \tag{1.13}$$

where $(w^*)^{K,+}$ stands for the trajectory in W_+ that follows step by step $w^* \in W_K^*$ from the time it first enters K. One then knows from proposition 1.3 of [8] that for $K \Subset \mathbb{Z}^d$:

$$\begin{aligned} &\mu_{K,u} \text{ is a Poisson point process on } (W_+, \mathcal{W}_+) \\ &\text{with intensity measure } u P_{e_K}, \end{aligned} \tag{1.14}$$

where the notation introduced above (1.2) is used. When $0 \leq u' < u$ and $K \Subset \mathbb{Z}^d$, one can define $\mu_{K,u',u}(\omega)$ for $\omega \in \Omega$, in analogy to (1.13), simply replacing the

inequality $u_i \leq u$ by the condition $u' < u_i \leq u$ in the formula for $\mu_{K,u}(\omega)$. One then finds that for $0 \leq u' < u$ and $K \Subset \mathbb{Z}^d$:

(1.15) $\mu_{K,u',u}$ and $\mu_{K,u'}$ are independent Poisson point processes on $(W_+, \mathcal{W}_+)$ with respective intensity measures $(u - u')P_{e_K}$ and $u'P_{e_K}$.

Given $\omega \in \Omega$, the interlacement at level $u \geq 0$ is the subset of $\mathbb{Z}^d$:

$$\mathcal{I}^u(\omega) = \bigcup_{u_i \leq u} \text{range}(w_i^*) \quad \text{if } \omega = \sum_{i \geq 0} \delta_{(w_i^*, u_i)} \tag{1.16}$$
$$= \bigcup_{K \Subset \mathbb{Z}^d} \bigcup_{w \in \text{supp}\, \mu_{K,u}(\omega)} w(\mathbb{N}),$$

where for $w^* \in W^*$, $\text{range}(w^*) = w(\mathbb{Z})$, for any $w \in W$, with $\pi^*(w) = w^*$, and the notation $\text{supp}\, \mu_{K,u}(\omega)$ refers to the support of the point measure $\mu_{K,u}(\omega)$. The vacant set at level u is then defined as

$$\mathcal{V}^u(\omega) = \mathbb{Z}^d \setminus \mathcal{I}^u(\omega) \quad \text{for } \omega \in \Omega,\ u \geq 0. \tag{1.17}$$

One then finds (see (1.54) of [8]) that

$$\mathcal{I}^u(\omega) \cap K = \bigcup_{w \in \text{supp}\, \mu_{K,u}(\omega)} w(\mathbb{N}) \cap K \quad \text{for } K \Subset \mathbb{Z}^d, u \geq 0, \omega \in \Omega, \tag{1.18}$$

and it follows with (1.14) that for $u \geq 0$,

$$\mathbb{P}[\mathcal{V}^u \supseteq K] = \exp\{-u\, \text{cap}(K)\} \quad \text{for all } K \Subset \mathbb{Z}^d, \tag{1.19}$$

a property that leads to the characterization by (0.1) of the law Q_u on $\{0, 1\}^{\mathbb{Z}^d}$ of the random subset $\mathcal{V}^u$ of $\mathbb{Z}^d$; see remark 2.2(2) of [8]. As mentioned in the introduction, Q_u is ergodic under spatial translations (cf. theorem 2.1 of [8]) and for $u > 0$, $\mathcal{I}^u(\omega)$ is $\mathbb{P}$-a.s. an infinite connected subset of $\mathbb{Z}^d$ (cf. corollary 2.3 of [8]). Intuitively it can be thought of as a "random fabric."

Remark 1.1. Since our principal objective is to prove that when $u > 0$ is small $\mathcal{V}^u$ percolates, it is important to point out that Q_u does not dominate any product of nondegenerate i.i.d. Bernoulli variables. Indeed, one knows from remark 2.5(2) of [8] that for $u > 0$ and $L \geq c(u)$ (see (0.1) below for the notation):

$$\mathbb{P}[\mathcal{I}^u \supseteq B(0, L)] = Q_u[Y_x = 0 \text{ for all } x \in B(0, L)] \tag{1.20}$$
$$\geq c \exp\{-cL^{d-2} \log L\}.$$

This shows that the probability that $\mathcal{V}^u \cap B(0, L) = \varnothing$ is rather "fat." In particular, the law Q_u of the indicator function of $\mathcal{V}^u$ cannot stochastically dominate (see [6, pp. 71–74]), the law of i.i.d. nondegenerate Bernoulli variables indexed by $\mathbb{Z}^d$. Note that when $d = 3$, the same argument even proves that the law of the indicator function of $\mathcal{V}^u \cap \mathbb{Z}^2$ cannot stochastically dominate the law of i.i.d. nondegenerate Bernoulli variables indexed by $\mathbb{Z}^2$. This rather high probability of absence of $\mathcal{V}^u$ in large boxes makes it difficult to prove that $\mathcal{V}^u$ percolates for small $u > 0$, with

a strategy based on dynamical or static renormalization in the spirit of chapter 7 in [3].

Incidentally, note that with (0.1) and (1.8) one also finds

$$\text{(1.21)}\quad \mathbb{P}[\mathcal{V}^u \supseteq B(0,L)] = Q_u[Y_x = 1 \text{ for all } x \in B(0,L)] \geq \exp\{-c\, L^{d-2}\},$$

and the probability that $\mathcal{V}^u$ covers $B(0,L)$ is rather "fat" as well, just as we conclude above that Q_u is not stochastically dominated by the law of nondegenerate i.i.d. Bernoulli variables indexed by $\mathbb{Z}^d$.

2 The Induction Step

In this section we develop a renormalization scheme that aims to show that the sequence of probabilities $q_n(u_n)$ that $\mathcal{I}^{u_n} \cap \mathbb{Z}^2$ contains a $*$-path between a given block of side length of order L_n and the complement of its L_n-neighborhood tends to 0. The sequence of length scales L_n, $n \geq 0$, grows rapidly to infinity; cf. (2.2), whereas u_n, $n \geq 0$, is a decreasing sequence of levels tending to $u_\infty > 0$; cf. (2.67). The heart of the matter in this section is the derivation of a recurrence relation enabling the control of $q_{n+1}(u_{n+1})$ in terms of $q_n(u_n)$; cf. (2.65). For this purpose we use a "sprinkling technique": trajectories of the interlacement with levels in $(u_{n+1}, u_n]$ are used to dominate the long-range interactions in the problem and restore some independence; see (2.61).

The main result of this section is Theorem 2.5. It reduces the task of proving a quantitative decay to zero of the sequence $q_n(u_n)$ to the question of being able to initiate the recurrence; see (0.11). Most of the work in the derivation of Theorem 2.5 is carried out in Proposition 2.1 and Proposition 2.4. We first need some notation.

We introduce a sequence of length scales as follows. We set

$$\text{(2.1)}\qquad a = \tfrac{1}{100},$$

and given $L_0 > 1$, define by induction

$$\text{(2.2)}\qquad L_{n+1} = \ell_n L_n \quad \text{for } n \geq 0 \text{ where } \ell_n = 100[L_n^a] + 1\ (\geq L_n^a).$$

We then introduce for each n a collection of pairwise disjoint d-dimensional boxes covering $\mathbb{Z}^2$, where we recall the convention introduced below (1.1). The index set of labels of boxes at level n is

$$\text{(2.3)}\qquad I_n = \{m = (n,i) : i \in \mathbb{Z}^2\} \quad \text{for } n \geq 0,$$

and to each $m = (n,i) \in I_n$ we attach the d-dimensional boxes

$$\text{(2.4)}\qquad C_m = \big([-L_n, L_n)^d + 2L_n i\big) \cap \mathbb{Z}^d,$$

$$\text{(2.5)}\qquad \tilde{C}_m = \bigcup_{m' \in I_n : d(C_{m'}, C_m) \leq 1} C_{m'} = \big([-3L_n, 3L_n)^d + 2L_n i\big) \cap \mathbb{Z}^d.$$

So for each $n \geq 0$, the boxes C_m, $m \in I_n$, are pairwise disjoint and their union covers $\mathbb{Z}^2$. As for $\widetilde{C}_m$ it is the union of C_m and the ℓ^∞-neighboring boxes of C_m at level n.

Note that one has the following "hierarchical" property: given $m \in I_{n+1}$, the trace of C_m on $\mathbb{Z}^2$ is partitioned by the respective traces on $\mathbb{Z}^d$ of the boxes at level n contained in C_m:

$$C_m \cap \mathbb{Z}^2 = \bigcup_{m' \in I_n : C_{m'} \subseteq C_m} C_{m'} \cap \mathbb{Z}^2 \quad \text{for } n \geq 0 \text{ and } m \in I_{n+1}. \tag{2.6}$$

Indeed, using the fact that ℓ_n is odd (cf. (2.2)), one simply writes

$$\begin{aligned}[-L_{n+1}, L_{n+1}) &= \bigcup_{k \text{ odd}, -\ell_n \leq k < \ell_n} [kL_n, (k+2)L_n) \\ &= \bigcup_{\ell \text{even}, -\ell_n < \ell < \ell_n} \big([-L_n, L_n) + \ell L_n\big),\end{aligned}$$

and inserts this identity "coordinatewise" into (2.4). Given $u \geq 0$, $n \geq 0$, and $m \in I_n$, one defines the event

$$B_m^u = \text{there is a } *\text{-path from } C_m \text{ to } \partial_{\text{int}} \widetilde{C}_m \text{ in } \mathcal{I}^u \cap \mathbb{Z}^2 \tag{2.7}$$

(we refer to the beginning of Section 1 for the notation). One also defines the probability

$$q_n(u) = \mathbb{P}[B_m^u] \quad \text{where } m \in I_n \text{ is arbitrary,} \tag{2.8}$$

and we have used the translation invariance of Q_u; see (0.1).

As already mentioned, we aim at deriving a bound of $q_{n+1}(u_{n+1})$ in terms of $q_n(u_n)$ along certain decreasing sequences u_n satisfying $u_\infty = \lim_n u_n > 0$. With this objective in mind, it is convenient to introduce for $n \geq 0$, the collections $\mathcal{K}_1$ and $\mathcal{K}_2$ of labels of boxes at level n contained in $\widetilde{C}_m$, where $m = (n+1, 0) \in I_{n+1}$, (i.e., C_m is the box at level $n+1$ containing the origin):

$$\begin{aligned}\mathcal{K}_1 &= \left\{ \begin{array}{l} m' \in I_n : m' = (n, i') \text{ where } i' = (i_1', i_2') \\ \text{with } \max(|i_1'|, |i_2'|) = \frac{\ell_n - 1}{2} \end{array} \right\} \\ &= \{m' \in I_n : C_{m'} \cap \partial_{\text{int}} C_m \neq \varnothing\},\end{aligned} \tag{2.9}$$

$$\begin{aligned}\mathcal{K}_2 &= \left\{ \begin{array}{l} m' \in I_n : m' = (n, i') \text{ where } i' = (i_1', i_2') \\ \text{with } \max(|i_1'|, |i_2'|) = \ell_n \end{array} \right\} \\ &= \{m' \in I_n : C_{m'} \cap S(0, 2L_{n+1}) \neq \varnothing\}.\end{aligned} \tag{2.10}$$

A key ingredient of the renormalization scheme comes from the following:

PROPOSITION 2.1 ($d \geq 3$) *For $L_0 \geq c$, all $n \geq 0$, and*

$$0 < u' \leq \left(1 + \frac{c_0}{\ell_n^{d-2}}\right)^{-1} u, \tag{2.11}$$

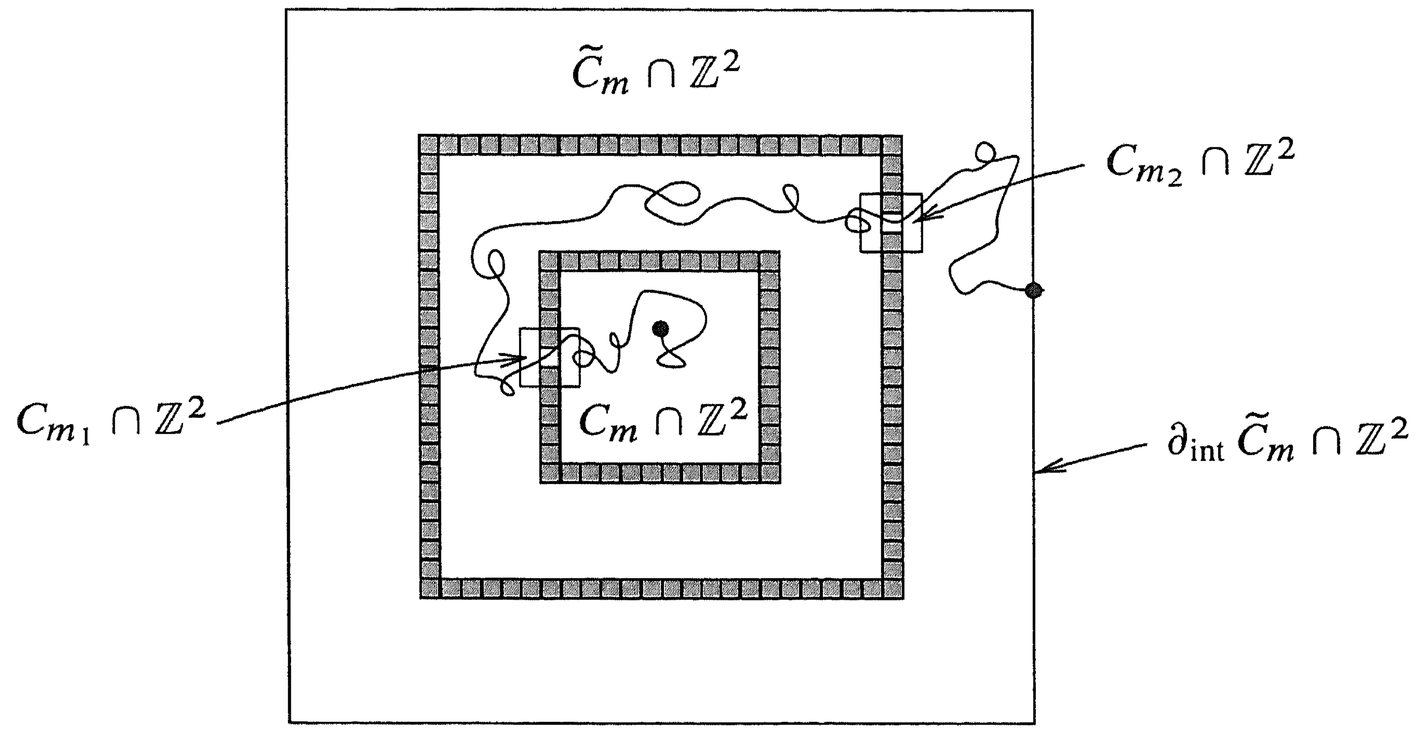

FIGURE 2.1. An illustration of the event B_m^u with a $*$-path in $\mathcal{I}^u \cap \mathbb{Z}^2$ from C_m to $\partial_{\text{int}} \tilde{C}_m$.

one has

$$q_{n+1}(u') \leq c_1 \ell_n^2 \big(q_n(u)^2 + u' L_n^{-2} + e^{-c_2(u-u')L_n^{d-2}}\big). \tag{2.12}$$

PROOF: We consider $n \geq 0$, $0 < u' < u$, and $m = (n+1, 0) \in I_{n+1}$, as above (2.9). Observe that any $*$-path in $\mathbb{Z}^2$ from C_m to $\partial_{\text{int}} \tilde{C}_m$ necessarily meets some C_{m_1} with m_1 in $\mathcal{K}_1$ and some C_{m_2} with m_2 in $\mathcal{K}_2$ and is neither contained in $\tilde{C}_{m_1}$ nor $\tilde{C}_{m_2}$ (see Figure 2.1), so that with a rough counting argument

$$q_{n+1}(u') \leq c \ell_n^2 \sup_{m_1, m_2} \mathbb{P}[B_{m_1}^{u'} \cap B_{m_2}^{u'}], \tag{2.13}$$

where the supremum runs over m_1 in $\mathcal{K}_1$ and m_2 in $\mathcal{K}_2$. Given such m_1 and m_2, we write

$$V = \tilde{C}_{m_1} \cup \tilde{C}_{m_2}, \tag{2.14}$$

and introduce the decomposition (in the notation of (1.13)):

$$\mu_{V,u} = \mu_{1,1} + \mu_{1,2} + \mu_{2,1} + \mu_{2,2}, \tag{2.15}$$

where for i, j distinct in $\{1, 2\}$ we have set

$$\begin{aligned} \mu_{i,j} &= \mathbb{1}_{\{X_0 \in \tilde{C}_{m_i},\, H_{\tilde{C}_{m_j}} < \infty\}} \mu_{V,u}, \\ \mu_{i,i} &= \mathbb{1}_{\{X_0 \in \tilde{C}_{m_i},\, H_{\tilde{C}_{m_j}} = \infty\}} \mu_{V,u}. \end{aligned} \tag{2.16}$$

Similarly, when $\mu_{V,u'}$ and $\mu_{V,u',u}$ (see (1.15) above) play the role of $\mu_{V,u}$, one obtains the identities (with hopefully obvious notation):

$$\mu_{V,u'} = \mu'_{1,1} + \mu'_{1,2} + \mu'_{2,1} + \mu'_{2,2}, \tag{2.17}$$

$$\mu_{V,u',u} = \mu^*_{1,1} + \mu^*_{1,2} + \mu^*_{2,1} + \mu^*_{2,2}, \tag{2.18}$$

together with

$$\mu_{i,j} = \mu'_{i,j} + \mu^*_{i,j} \quad \text{for } 1 \le i, j \le 2. \tag{2.19}$$

In view of (1.15) we also see that

(2.20) $\mu'_{i,j}, \mu^*_{i,j}, 1 \le i, j \le 2$, are independent Poisson point processes on W_+.

Roughly speaking the possible dependence between $B^{u'}_{m_1}$ and $B^{u'}_{m_2}$ on the right-hand side of (2.13) originates from the contribution of $\mu'_{1,2} + \mu'_{2,1}$ (see (2.17)), when one considers the trace on $V \cap \mathbb{Z}^2$ of the trajectories in the support of $\mu_{V,u'}$. In essence our strategy is to exhibit some "domination" of the trace on $V \cap \mathbb{Z}^2$ of trajectories in the support of $\mu'_{1,2} + \mu'_{2,1}$ in terms of the corresponding trace of trajectories in the support of $\mu^*_{1,1} + \mu^*_{2,2}$ and "correction terms" when u is sufficiently bigger than u'. This is the sprinkling technique, and it will produce a decoupling and a natural control of $q_{n+1}(u')$ in terms of $q_n(u)^2$ thanks to (2.19) and (2.20).

We now introduce the ℓ^∞-neighborhood of size $[L_{n+1}/10]$ of $V = \tilde{C}_{m_1} \cup \tilde{C}_{m_2}$:

$$U = \left\{ z \in \mathbb{Z}^d : d(z, V) \le \frac{L_{n+1}}{10} \right\}. \tag{2.21}$$

This set is the union of two disjoint boxes that are translates of $[-R, R)^d$, with $R = 3L_n + [L_{n+1}/10]$, and the ℓ^∞-norm of the vector translating one to the other box is at least L_{n+1}.

For a trajectory in W_+, the times R_k, $k \ge 1$, of successive returns to V, and D_k, $k \ge 1$, of successive departures from U are defined as (see (1.2) for the notation)

$$\begin{aligned} &R_1 = H_V, && D_1 = T_U \circ \theta_{R_1} + R_1, \\ &R_{k+1} = R_1 \circ \theta_{D_k} + D_k, && D_{k+1} = D_1 \circ \theta_{D_k} + D_k \quad \text{for } k \ge 1, \end{aligned} \tag{2.22}$$

so that $0 \le R_1 \le D_1 \le \cdots \le R_k \le D_k \le \cdots \le \infty$.

To control the dependence between the events in the right-hand side of (2.13), we consider $r \ge 2$ and write

$$\mu'_{1,2} + \mu'_{2,1} = \sum_{2 \le \ell \le r} \rho'_\ell + \overline{\rho}, \tag{2.23}$$

where

$$\rho'_\ell = \mathbb{1}_{\{R_\ell < \infty = R_{\ell+1}\}} (\mu'_{1,2} + \mu'_{2,1}) \quad \text{for } \ell \ge 1 \tag{2.24}$$

(note that $\rho'_1 = 0$, since $(\mu'_{1,2} + \mu'_{2,1})$-a.s., $R_1 = 0$, and $R_2 < \infty$), and

$$\overline{\rho} = \mathbb{1}_{\{R_{r+1} < \infty\}} (\mu'_{1,2} + \mu'_{2,1}) \tag{2.25}$$

($\overline{\rho}$ will be treated as a correction term; see (2.30) and (2.61) below). As as result of (2.20) we see that

(2.26) ρ'_ℓ, $2 \le \ell \le r$, $\overline{\rho}$, $\mu'_{1,1}$, and $\mu'_{2,2}$ are independent Poisson point processes on W_+.

We denote by ξ'_ℓ the intensity measure of ρ'_ℓ and by $\overline{\xi}$ the intensity measure of $\overline{\rho}$. From the analogues of (2.16) and (1.14) with u' in place of u, we infer that

$$\xi'_\ell = u' P_{e_V}[E, R_\ell < \infty = R_{\ell+1}, \cdot], \ 2 \le \ell \le r, \tag{2.27}$$

and that

$$\overline{\xi} = u' P_{e_V}[E, R_{r+1} < \infty, \cdot], \tag{2.28}$$

where $E = \{X_0 \in \widetilde{C}_{m_1}, \ H_{\widetilde{C}_{m_2}} < \infty\} \cup \{X_0 \in \widetilde{C}_{m_2}, \ H_{\widetilde{C}_{m_1}} < \infty\}$.

We first bound the total mass $\overline{\xi}(W_+)$ of $\overline{\xi}$. This is performed in an analogous fashion to (3.23)–(3.25) of [8]. With (1.6), (1.8), and classical bounds on the Green function (cf. [5], p. 31), we find that

$$\sup_{x \in U^c} P_x[H_V < \infty] \le \frac{c_3}{\ell_n^{d-2}}. \tag{2.29}$$

Discarding the event E in (2.28) and applying the strong Markov property at times $D_r, D_{r-1}, \ldots, D_1$, we find that

$$\overline{\xi}(W_+) \le c u' L_n^{d-2} \left(\frac{c_3}{\ell_n^{d-2}} \right)^r \overset{(2.2)}{\le} c c_3^r u' L_n^{(d-2)(1-ar)}, \tag{2.30}$$

where we used the subadditivity property of the capacity (see (1.5) below), and the right-hand inequality of (1.8) when bounding $\mathrm{cap}(V)$; see (2.14).

We then turn our attention to the point measures ρ'_ℓ. We introduce the set of finite paths (see the beginning of Section 1 for the definition)

$$\begin{aligned} \mathcal{T} = \{w = (w(i))_{0 \le i \le N} \text{ finite path; } w(0) \in V, \\ w(N) \in \partial U, \text{ and } w(i) \in U \text{ for } 0 \le i < N\}. \end{aligned} \tag{2.31}$$

Given $\ell \ge 1$, we define the map ϕ_ℓ from $\{R_\ell < \infty = R_{\ell+1}\} \subseteq W_+$ into $\mathcal{T}^\ell$ such that

$$\begin{aligned} w \to \phi_\ell(w) = (w_1, \ldots, w_\ell) \\ \text{with } w_k(\cdot) = (X_{R_k + \cdot}(w))_{0 \le \cdot \le D_k(w) - R_k(w)} \quad \text{for } 1 \le k \le \ell. \end{aligned} \tag{2.32}$$

The next lemma yields a control on the probability that the walk hits $y \in V$ at its entrance in V when it starts in $\partial U \cup \partial_{\mathrm{int}} U$.

LEMMA 2.2

$$\sup_{z \in \partial U \cup \partial_{\mathrm{int}} U} P_z[H_V < \infty, \ X_{H_V} = y] \le \frac{c}{L_{n+1}^{d-2}} e_V(y) \quad \textit{for all } y \in V. \tag{2.33}$$

PROOF: From the inclusion $V \subseteq U$, one deduces the identity

$$e_V(y) = P_{e_U}[H_V < \infty, \ X_{H_V} = y] \quad \text{for all } y \in V. \tag{2.34}$$

This "sweeping" identity can be seen as the consequence of (1.46) of [8] for the intensity measures of the Poisson point processes under consideration there. Defining for $y \in V$ the nonnegative harmonic function on V^c

$$\psi(z) = P_z[H_V < \infty, \ X_{H_V} = y],$$

one finds with the left-hand inequality of (1.8) and (2.34) that

$$e_V(y) \geq c L_{n+1}^{d-2} \inf_{z \in \partial_{\text{int}} U} \psi(z). \tag{2.35}$$

With the Harnack inequality (cf. [5, p. 42]) and a covering argument of $\partial U \cup \partial_{\text{int}} U$ by finitely many balls in V^c with radius cL_{n+1} and centers in $\partial_{\text{int}} U$, one knows that

$$\sup_{z \in \partial U \cup \partial_{\text{int}} U} \psi(z) \leq c \inf_{z \in \partial U \cup \partial_{\text{int}} U} \psi(z). \tag{2.36}$$

Inserting this inequality in (2.35) yields for $y \in V$

$$e_V(y) \geq c L_{n+1}^{d-2} \sup_{z \in \partial U \cup \partial_{\text{int}} U} P_z[H_V < \infty,\ X_{H_V} = y],$$

and hence (2.33). □

For $2 \leq \ell \leq r$, we can view ρ'_ℓ in (2.24) as a point process on $\{R_\ell < \infty = R_{\ell+1}\}$ $(\subseteq W_+)$ and then introduce

$$\widetilde{\rho}'_\ell \text{ the image of } \rho'_\ell \text{ under } \phi_\ell. \tag{2.37}$$

Thus with (2.26) we find that

$$\widetilde{\rho}'_\ell,\ 2 \leq \ell \leq r,\ \overline{\rho},\ \mu'_{1,1}, \text{ and } \mu'_{2,2} \text{ are independent Poisson point processes.} \tag{2.38}$$

We denote by $\widetilde{\xi}'_\ell$ the intensity of $\widetilde{\rho}'_\ell$ for $2 \leq \ell \leq r$. In view of (2.27) we see that for $w_1, \ldots, w_\ell$ in $\mathcal{T}$,

$$\widetilde{\xi}'_\ell(w_1, \ldots, w_\ell) = u' P_{e_V}[E,\ R_\ell < \infty = R_{\ell+1}, \\ (X_{R_k + \cdot})_{0 \leq \cdot \leq D_k - R_k} = w_k(\cdot),\ 1 \leq k \leq \ell]. \tag{2.39}$$

We will now prove with the help of Lemma 2.2 that for $2 \leq \ell \leq r$,

$$\widetilde{\xi}'_\ell \leq c_4 u' L_n^{d-2} \left(\frac{c_5}{\ell_n^{d-2}} \right)^{\ell-1} P_{\overline{e}_V}[(X_\cdot)_{0 \leq \cdot \leq T_U} \in \cdot]^{\otimes \ell}, \tag{2.40}$$

where $\overline{e}_V$ stands for the normalized equilibrium measure of V,

$$\overline{e}_V = \frac{1}{\text{cap}(V)} e_V. \tag{2.41}$$

Indeed, observe that for $w_1, \ldots, w_\ell$ in $\mathcal{T}$, in view of (2.39), by discarding the event E and using the strong Markov property, we find that

$$\begin{aligned}
&\widetilde{\xi}'_\ell(w_1, \ldots, w_\ell) \\
&\leq \quad u' E_{e_V}\big[R_\ell < \infty, (X_{R_k+\cdot})_{0\leq\cdot\leq D_k - R_k} = w_k(\cdot),\ 1 \leq k < \ell, \\
&\qquad\qquad P_{X_{R_\ell}}[(X_\cdot)_{0\leq\cdot\leq T_U} = w_\ell(\cdot)]\big] \\
&= \quad u' E_{e_V}\big[D_{\ell-1} < \infty,\ (X_{R_k+\cdot})_{0\leq\cdot\leq D_k - R_k} = w_k(\cdot),\ 1 \leq k < \ell, \\
&\qquad\qquad E_{X_{D_{\ell-1}}}[H_V < \infty, P_{X_{H_V}}[(X_\cdot)_{0\leq\cdot\leq T_U} = w_\ell(\cdot)]]\big] \\
&\overset{(2.33)}{\leq} \quad u' P_{e_V}\big[R_{\ell-1} < \infty, (X_{R_k+\cdot})_{0\leq\cdot\leq D_k - R_k} = w_k(\cdot), 1 \leq k < \ell\big] \\
&\qquad\qquad \frac{c}{L_{n+1}^{d-2}} P_{e_V}\big[(X_\cdot)_{0\leq\cdot\leq T_U} = w_\ell(\cdot)\big] \\
&\overset{\text{induction}}{\leq} u'\Big(\frac{c}{L_{n+1}^{d-2}}\Big)^{\ell-1} \prod_{k=1}^{\ell} P_{e_V}\big[(X_\cdot)_{0\leq\cdot\leq T_U} = w_k(\cdot)\big].
\end{aligned} \tag{2.42}$$

Claim (2.40) now follows by bounding $\mathrm{cap}(V)$ with (1.8) and the subadditive property below (1.5) and using (2.41).

We can also view $\mathbb{1}_{\{R_2=\infty\}}(\mu^*_{1,1} + \mu^*_{2,2})$ as a Poisson point process on $\{R_1 < \infty = R_2\}$ $(\subseteq W_+)$ (note that $R_1 = 0$, $\mu^*_{1,1} + \mu^*_{2,2}$-a.s. in view of the analogue of (2.16) for $\mu^*_{i,j}$). We then introduce the Poisson point process on $\mathcal{T}$:

$$\widetilde{\rho}^*_1 \text{ the image of } \mathbb{1}_{\{R_2=\infty\}}(\mu^*_{1,1} + \mu^*_{2,2}) \text{ under } \phi_1, \tag{2.43}$$

and denote by $\widetilde{\xi}^*_1$ its intensity measure (a measure on $\mathcal{T}$). In view of (1.15), (2.18), and the analogue of (2.16) for $\mu^*_{i,j}$, we find that

$$\begin{aligned}
\widetilde{\xi}^*_1(w) = (u - u')\, P_{e_V}[(X_\cdot)_{0\leq\cdot\leq T_U} = w(\cdot),\ H_V \circ \theta_{T_U} = \infty] \\
\text{for } w \in \mathcal{T}.
\end{aligned} \tag{2.44}$$

As a result of (2.29), we see that for $L_0 \geq c$,

$$\inf_{x\in\partial U} P_x[H_V = \infty] \geq \tfrac{1}{2}, \tag{2.45}$$

and hence with (1.8) and the strong Markov property applied at time T_U to the probability in (2.44), we deduce that

$$\widetilde{\xi}^*_1 \geq c_6(u - u') L_n^{d-2} P_{\overline{e}_V}[(X_\cdot)_{0\leq\cdot\leq T_U} \in \cdot\,]. \tag{2.46}$$

The upper bounds (2.40) and the lower bound (2.46) will be our main instrument when seeking to dominate the trace on $V \cap \mathbb{Z}^2$ of trajectories in the support of $\mu'_{1,2} + \mu'_{2,1}$ in terms of the trace on $\mathbb{Z}^2 \cap V$ of trajectories in the support of

$\mu^*_{1,1} + \mu^*_{2,2}$. With this goal in mind, we introduce the random subsets of $V \cap \mathbb{Z}^2$:

$$\mathcal{I}'_{i,i} = V \cap \mathbb{Z}^2 \cap \Big(\bigcup_{w \in \mathrm{supp}(\mu'_{i,i})} \mathrm{range}(w) \Big) \quad \text{for } i = 1, 2,$$

$$\text{(2.47)} \qquad \widetilde{\mathcal{I}}'_\ell = V \cap \mathbb{Z}^2 \cap \Big(\bigcup_{(w_1,\dots,w_\ell) \in \mathrm{supp}\, \widetilde{\rho}'_\ell} \mathrm{range}(w_1) \cup \cdots \cup \mathrm{range}(w_\ell) \Big)$$
$$\text{for } 2 \le \ell \le r,$$

$$\overline{\mathcal{I}} = V \cap \mathbb{Z}^2 \cap \Big(\bigcup_{w \in \mathrm{supp}\, \overline{\rho}} \mathrm{range}(w) \Big).$$

Note that with (2.38), it follows that

$$\text{(2.48)} \qquad \mathcal{I}'_{1,1},\ \mathcal{I}'_{2,2},\ \widetilde{\mathcal{I}}'_\ell,\ 2 \le \ell \le r, \text{ and } \overline{\mathcal{I}} \text{ are independent under } \mathbb{P}.$$

Moreover, in view of (2.17), (2.23), and (2.37), one has the identity

$$\text{(2.49)} \qquad \mathcal{I}^{u'} \cap V \cap \mathbb{Z}^2 = \mathcal{I}'_{1,1} \cup \mathcal{I}'_{2,2} \cup \Big(\bigcup_{2 \le \ell \le r} \widetilde{\mathcal{I}}'_\ell \Big) \cup \overline{\mathcal{I}}.$$

In a similar fashion to (2.47), we can define

$$\text{(2.50)} \qquad \mathcal{I}^*_{i,i} = V \cap \mathbb{Z}^2 \cap \Big(\bigcup_{w \in \mathrm{supp}\, \mu^*_{i,i}} \mathrm{range}(w) \Big) \quad \text{for } i = 1, 2,$$
$$\mathcal{I}^* = V \cap \mathbb{Z}^2 \cap \Big(\bigcup_{w \in \mathrm{supp}\, \widetilde{\rho}^*_1} \mathrm{range}(w) \Big).$$

Taking (2.20) and (2.43) into account, we see that

$$\text{(2.51)} \qquad \mathcal{I}^*,\ \mathcal{I}'_{1,1},\ \mathcal{I}'_{2,2},\ \widetilde{\mathcal{I}}'_\ell,\ 2 \le \ell \le r, \text{ and } \overline{\mathcal{I}} \text{ are independent under } \mathbb{P},$$

and further that

$$\text{(2.52)} \qquad \mathcal{I}^* \subseteq \mathcal{I}^*_{1,1} \cup \mathcal{I}^*_{2,2}.$$

As we now explain, we will construct a coupling of $\widetilde{\mathcal{I}}'_\ell$, $2 \le \ell \le r$, and $\mathcal{I}^*$. We consider on some auxiliary probability space independent Poisson variables, N'_ℓ, $2 \le \ell \le r$, and N^*_ℓ, $1 \le \ell \le r$, with respective intensities (cf. (2.40) and (2.46))

$$\text{(2.53)} \qquad \lambda'_\ell = c_4 u' L_n^{d-2} \Big(\frac{c_5}{\ell_n^{d-2}} \Big)^{\ell - 1}, \qquad 2 \le \ell \le r, \quad \text{and}$$
$$\lambda^*_\ell = \frac{c_6}{r} (u - u') L_n^{d-2}, \qquad 1 \le \ell \le r,$$

as well as i.i.d. $\mathcal{T}$-valued variables γ^ℓ_i, $1 \le \ell \le r$, $i \ge 1$, independent of the N'_ℓ, $2 \le \ell \le r$, N^*_ℓ, $1 \le \ell \le r$, with common distribution $P_{\overline{e}_V}[(X_\cdot)_{0 \le \cdot \le T_U} \in \cdot\,]$.

We then define the point processes:

$$\Gamma'_\ell = \sum_{1\le i\le N'_\ell} \delta_{(\gamma^\ell_{1+(i-1)\ell}, \gamma^\ell_{2+(i-1)\ell}, \dots, \gamma^\ell_{\ell+(i-1)\ell})} \quad \text{on } \mathcal{T}^\ell \text{ with } 2\le \ell \le r, \tag{2.54}$$

and

$$\Gamma^*_\ell = \sum_{1\le i\le N^*_\ell} \delta_{\gamma^\ell_i} \quad \text{on } \mathcal{T} \text{ with } 1\le \ell \le r. \tag{2.55}$$

We thus find that

$$\Gamma'_\ell,\ 2\le \ell \le r, \text{ are independent Poisson point processes with respective intensity measures } \lambda'_\ell P_{\overline{e}_V}[(X_\cdot)_{0\le\cdot\le T_U} \in \cdot]^{\otimes \ell} \overset{(2.40)}{\ge} \widetilde{\xi}'_\ell), \text{ and} \tag{2.56}$$

$$\Gamma^*_\ell,\ 1\le \ell \le r, \text{ are independent Poisson point processes with identical intensity measure } \lambda^*_\ell P_{\overline{e}_V}[(X_\cdot)_{0\le\cdot\le T_U} \in \cdot] \overset{(2.46)}{\le} \tfrac{1}{r}\widetilde{\xi}^*_1). \tag{2.57}$$

In view of (2.56) and (2.57), we can thus construct on some probability space $(\Sigma, \mathcal{F}, Q)$ a coupling of $\widetilde{\rho}'_\ell$, $\widetilde{N}'_\ell$, Γ'_ℓ, $2\le\ell\le r$, $\widetilde{\rho}^*_1$, and N^*_ℓ, Γ^*_ℓ, $1\le\ell\le r$, so that

$$\widetilde{\rho}'_\ell \le \Gamma'_\ell \text{ for } 2\le\ell\le r \quad \text{and} \quad \sum_{1\le\ell\le r} \Gamma^*_\ell \le \widetilde{\rho}^*_1 \tag{2.58}$$

(for instance, in view of the inequality in the last line of (2.56) we construct the law of $\widetilde{\rho}'_\ell$ by thinning Γ'_ℓ, and in view of the inequality in the last line of (2.57), we construct the laws of $\widetilde{\rho}^*_1$ by adding an independent $\mathcal{T}$-valued Poisson point process).

From the definition of $\widetilde{\mathcal{I}}'_\ell$ in (2.47) and $\mathcal{I}^*$ in (2.50), it then follows that on the event $\bigcap_{2\le\ell\le r}\{N^*_\ell \ge r\, N'_\ell\}$, one has

$$\begin{aligned}
\bigcup_{2\le\ell\le r} \widetilde{\mathcal{I}}'_\ell &\overset{(2.54)}{\underset{(2.58)}{\subseteq}} V\cap\mathbb{Z}^2 \\
&\qquad \cap\Big(\bigcup_{2\le\ell\le r}\Big(\bigcup_{1\le i\le N'_\ell} \operatorname{range}(\gamma^\ell_{1+(i-1)\ell}) \cup\cdots\cup \operatorname{range}(\gamma^\ell_{\ell+(i-1)\ell})\Big)\Big) \\
&\subseteq V\cap\mathbb{Z}^2\cap\Big(\bigcup_{2\le\ell\le r}\Big(\bigcup_{1\le j\le N^*_\ell} \operatorname{range}(\gamma^\ell_j)\Big)\Big) \\
&\overset{(2.55)}{\underset{(2.58)}{\subseteq}} V\cap\mathbb{Z}^2\cap\Big(\bigcup_{w\in\operatorname{supp}\widetilde{\rho}^*_1} \operatorname{range}(w)\Big) = \mathcal{I}^*.
\end{aligned} \tag{2.59}$$

Hence the coupling we constructed on $(\Sigma, \mathcal{F}, Q)$ leads to the bound

$$Q\Big(\mathcal{I}^* \supseteq \bigcup_{2\le\ell\le r} \widetilde{\mathcal{I}}'_\ell\Big) \ge 1 - \sum_{2\le\ell\le r} Q(N^*_\ell < rN'_\ell). \tag{2.60}$$

This inequality plays a pivotal role in the sprinkling technique we employ in order to control interactions. Indeed, we can bound the probability in the right-hand side of (2.13) as follows:

$$
\begin{aligned}
&\mathbb{P}[B^{u'}_{m_1} \cap B^{u'}_{m_2}] \\
&\overset{(2.49)}{=} \mathbb{P}\big[\text{there are } *\text{-paths from } C_{m_1} \text{ to } \partial_{\text{int}}\widetilde{C}_{m_1} \text{ and from } C_{m_2} \\
&\qquad \text{to } \partial_{\text{int}}\widetilde{C}_{m_2} \text{ in } \mathcal{I}'_{1,1} \cup \mathcal{I}'_{2,2} \cup \big(\textstyle\bigcup_{2\le\ell\le r} \widetilde{\mathcal{I}}'_\ell\big) \cup \overline{\mathcal{I}}\big] \\
&\overset{(2.51)}{\underset{(2.60)}{\le}} \mathbb{P}\big[\text{there are } *\text{-paths from } C_{m_1} \text{ to } \partial_{\text{int}}\widetilde{C}_{m_1} \text{ and from } C_{m_2} \\
&\qquad \text{to } \partial_{\text{int}}\widetilde{C}_{m_2} \text{ in } \mathcal{I}'_{1,1} \cup \mathcal{I}'_{2,2} \cup \mathcal{I}^* \cup \overline{\mathcal{I}}\big] \\
&\qquad + \sum_{2\le\ell\le r} Q(N^*_\ell < r\, N'_\ell) \qquad (2.61) \\
&\overset{(2.20)}{\underset{(2.52)}{\le}} \mathbb{P}\big[\text{there is a } *\text{-path from } C_{m_1} \text{ to } \partial_{\text{int}}\widetilde{C}_{m_1} \text{ in } \mathcal{I}'_{1,1} \cup \mathcal{I}^*_{1,1}\big] \\
&\qquad \mathbb{P}\big[\text{there is a } *\text{-path from } C_{m_2} \text{ to } \partial_{\text{int}}\widetilde{C}_{m_2} \text{ in } \mathcal{I}'_{2,2} \cup \mathcal{I}^*_{2,2}\big] \\
&\qquad + \mathbb{P}[\overline{\mathcal{I}} \ne \varnothing] + \sum_{2\le\ell\le r} Q(N^*_\ell < r\, N'_\ell) \\
&\overset{(2.8)}{\underset{(2.19)}{\le}} q_n(u)^2 + \overline{\xi}(W_+) + \sum_{2\le\ell\le r} Q(N^*_\ell < r\, N'_\ell),
\end{aligned}
$$

where we have used the fact that $(\mathcal{I}'_{i,i} \cup \mathcal{I}^*_{i,i}) \cap \widetilde{C}_{m_j} = \varnothing$ for $1 \le i \ne j \le 2$, to decouple probabilities after the second inequality and bounded $\mathbb{P}[\overline{\mathcal{I}} \ne \varnothing]$ by $\mathbb{P}[\overline{\rho} \ne 0] \le \overline{\xi}(W_+)$ in the last inequality.

We will now bound the last term in the last line of (2.61). To this effect we ensure that (see (2.53))

$$
\lambda^*_\ell \ge 4r\lambda'_\ell \quad \text{for } 2 \le \ell \le r \tag{2.62}
$$

by imposing $L_0 \ge c$ and

$$
u - u' \ge c'_0 \frac{r^2}{\ell_n^{d-2}}\, u' \tag{2.63}
$$

(with the choice (2.66) below, this will yield (2.11)).

As result of (2.62) we thus find that for $2 \le \ell \le r$,

$$
\begin{aligned}
Q(N^*_\ell < rN'_\ell) &\le Q\left(N^*_\ell \le \frac{\lambda^*_\ell}{2}\right) + Q\left(N'_\ell \ge \frac{\lambda^*_\ell}{2r}\right) \\
&\le ce^{-\frac{c}{r}\lambda^*_\ell} \overset{(2.53)}{=} ce^{-\frac{c}{r^2}(u-u')L_n^{d-2}},
\end{aligned} \tag{2.64}
$$

where we have dominated N'_ℓ by a Poisson variable of intensity $\lambda^*_\ell/4r$ thanks to (2.62), and used classical exponential bounds on the tail of Poisson variables in the second inequality.

Coming back to (2.13), we see that when $L_0 \geq c$ for $n \geq 0$ and $r \geq 2$, when (2.63) holds, one finds by collecting (2.30), (2.61), and (2.64) that

$$q_{n+1}(u') \leq c\ell_n^2\big(q_n(u)^2 + u' c_3^r L_n^{(d-2)(1-ar)} + re^{-\frac{c}{r^2}(u-u')L_n^{d-2}}\big). \tag{2.65}$$

We can now choose

$$r = \frac{3}{a} \overset{(2.1)}{=} 300, \tag{2.66}$$

and with this choice (2.65) is more than enough to yield the claim (2.12). □

Remark 2.3.

(1) It is clear from the above proof that by adjusting the choice of r in (2.66), we can produce an arbitrary negative power of L_n in place of L_n^{-2} in the right-hand side of (2.12) (of course, we adjust constants there as well). The present choice will suffice for our purposes.

(2) The inequality (2.65) and the auxiliary condition (2.63) play a similar role to (3.52) and (3.45) of [8] (which pertain to the control of crossings of the vacant set and are later applied to increasing sequences u_n that remain bounded).

Let us give some comments. Roughly speaking, in [8] one dominates the trace on the single box $\widetilde{C}_{m_2}$ of trajectories in $\mu'_{1,2} + \mu'_{2,1}$ in terms of traces on $\widetilde{C}_{m_2}$ of $\mu^*_{2,2}$. In the present work we must handle both boxes at once when bounding the probability in the right-hand side of (2.13). We dominate the simultaneous traces on $\widetilde{C}_{m_1} \cap \mathbb{Z}^2$ and $\widetilde{C}_{m_2} \cap \mathbb{Z}^2$ of $\mu'_{1,2} + \mu'_{2,1}$ in terms of the independent traces of $\mu^*_{1,1}$ on $\widetilde{C}_{m_1} \cap \mathbb{Z}^2$ and of $\mu^*_{2,2}$ on $\widetilde{C}_{m_2} \cap \mathbb{Z}^2$.

Whereas in [8] one specifies the parameter r relatively late, depending on the successive choices of L_0 and u_0, in order to cope with the increasing sequences u_n and the indeterminacy of u_0, in the present context r is simply fixed by the choice (2.66). However, the last term of (2.65) is specific to the present control of a Poisson point process affecting two boxes in terms of two independent Poisson point processes concerning each respective box. Quite naturally this term deteriorates when the intensity λ^*_ℓ in (2.53) becomes small.

We will now see how one can propagate controls on the probabilities $q_n(u_n)$ (cf. (2.8)) and choose sequences u_n that decrease not too fast so that $u_\infty = \lim u_n > 0$ but still sufficiently fast so that $u_n - u_{n+1}$ is large enough. The last term in the right-hand side of (2.12) suggests picking u_{n+1} sufficiently smaller than u_n in (2.67) below, and not trying to saturate (2.11) (with u_{n+1} and u_n, respectively, playing the roles of u' and u).

Given u_0 in $(0, 1]$ and $L_0 > 1$, we define the sequence u_n, $n \geq 0$, via

$$u_{n+1} = \Big(1 + \frac{1}{\log L_n}\Big)^{-1} u_n \quad \text{for } n \geq 0. \tag{2.67}$$

We now derive a crucial propagation of certain controls from one scale to the next. We refer to Proposition 2.1 for notation.

PROPOSITION 2.4 ($d \geq 3$) *When* $L_0 \geq c$ *and* $0 < u_0 \leq 1$, *if for some* $n \geq 0$,

$$c_2(u_n - u_{n+1})L_n^{d-2} \geq 2\log L_n, \tag{2.68a}$$

$$a_n \overset{\text{def}}{=} c_1 \ell_n^2 q_n(u_n) \leq L_n^{-1}, \tag{2.68b}$$

then (2.68a) *and* (2.68b) *hold true with* $n+1$ *in place of* n.

PROOF: Observe that when $L_0 \geq c$,

$$u_n \overset{(2.67)}{=} \left(1 + \frac{1}{\log L_n}\right) u_{n+1} \geq \left(1 + \frac{c_0}{\ell_n^{d-2}}\right) u_{n+1},$$

and hence with (2.12) where we set $u = u_n$ and $u' = u_{n+1}$, noting that $u_{n+1} \leq u_0 \leq 1$, we find that

$$a_{n+1} \leq \left(\frac{\ell_{n+1}}{\ell_n}\right)^2 a_n^2 + c(\ell_{n+1}\ell_n)^2 (L_n^{-2} + e^{-c_2(u_n - u_{n+1})L_n^{d-2}}). \tag{2.69}$$

We will now check (2.68a) at level $n+1$. For $L_0 \geq c$, we find that

$$1 \leq \log L_k \leq \log L_{k+1} \overset{(2.2)}{\leq} (1+a)\log L_k + c \leq 2\log L_k$$

for all $k \geq 0$, and therefore when $L_0 \geq c'$,

$$\begin{aligned} c_2(u_{n+1} - u_{n+2})L_{n+1}^{d-2} &\overset{(2.2)}{\underset{(2.67)}{=}} c_2 \frac{u_{n+2}}{\log L_{n+1}} L_n^{d-2}\ell_n^{d-2} \\ &\geq \frac{c_2}{4} \frac{u_{n+1}}{\log L_n} L_n^{d-2}\ell_n^{d-2} \\ &= c_2(u_n - u_{n+1})L_n^{d-2}\frac{\ell_n^{d-2}}{4} \\ &\overset{(2.68a)}{\geq} \frac{1}{2}\log L_n \ell_n^{d-2} \\ &\geq \frac{1}{4}\log L_{n+1}\ell_n^{d-2} \geq 2\log L_{n+1}. \end{aligned} \tag{2.70}$$

This shows that (2.68a) holds at level $n+1$.

We now check (2.68b) at level $n+1$. We note that

$$\frac{\ell_{n+1}}{\ell_n} \overset{(2.2)}{\leq} c\frac{L_n^{(a+1)a}}{L_n^a} = cL_n^{a^2} \quad \text{and} \quad \ell_{n+1}\ell_n \leq cL_n^{2a+a^2}.$$

We thus see that when $L_0 \geq c$, (2.69) and the induction hypothesis yield that

$$a_{n+1} \leq cL_n^{2a^2}a_n^2 + cL_n^{4a+2a^2}(L_n^{-2} + L_n^{-2}) \leq cL_n^{4a+2a^2-2} \overset{(2.1)}{\leq}$$
$$L_{n+1}^{-1}\frac{L_{n+1}}{L_n}cL_n^{6a-1} \overset{(2.2)}{\leq} L_{n+1}^{-1}cL_n^{7a-1} \leq L_{n+1}^{-1}.$$

This shows (2.68b) at level $n+1$ and completes the proof of Proposition 2.4. □

With Proposition 2.4, obtaining a control on the sequence of probabilities $q_n(u_n)$ is reduced to initiating the recurrence in (2.68). We collect the results we will need for the next section in the following theorem (see Proposition 2.1 and (2.67) for the notation).

THEOREM 2.5 ($d \geq 3$) *When $L_0 \geq c_7$, if for the choice*

$$u_0 = \frac{4}{c_2} (\log L_0)^2 L_0^{-(d-2)} \tag{2.71}$$

it holds that

$$c_1 \ell_0^2 q_0(u_0) \leq L_0^{-1}, \tag{2.72}$$

then

$$c_1 \ell_n^2 q_n(u_\infty) \leq c_1 \ell_n^2 q_n(u_n) \leq L_n^{-1} \quad \text{for all } n \geq 0, \tag{2.73}$$

where $u_\infty \overset{\text{def}}{=} u_0 \times \prod_{n \geq 0} (1 + 1/\log L_n)^{-1} \in (0, 1]$.

PROOF: We see that for $L_0 \geq c_7$ with the choice in (2.71), $0 < u_0 \leq 1$, and also that

$$c_2(u_0 - u_1) L_0^{d-2} = c_2 \frac{u_1}{\log L_0} L_0^{d-2} \geq \frac{c_2}{2} \frac{u_0}{\log L_0} L_0^{d-2} = 2 \log L_0.$$

Claim (2.73) now follows from Proposition 2.4, and the positivity of u_∞ is a consequence of the inequality $L_n \geq L_0^{(1+a)^n}$ (cf. (2.2)). □

Remark 2.6.

(1) It should be realized that initiating the induction scheme we have developed in this section, i.e., checking (2.72) for $L_0 \geq c_7$ and u_0 as in (2.71), is not a mere formality. To illustrate the point, observe that one can replace $\mathbb{Z}^2$ with $\mathbb{Z}^d$ in (2.3) and (2.7); i.e., consider instead boxes of size of order L_n filling the whole space $\mathbb{Z}^d$ and the event that such a box is connected by a $*$-path in $\mathcal{I}^u$ to the complement of the neighborhood of size of order L_n corresponding to (2.5) in this modified setup.

Then small variations of the arguments used in Propositions 2.1 and 2.4, with the modified choice $a = 1/100d$ in (2.1), will yield a similar result to Theorem 2.5, with possibly different constants and the multiplicative factor ℓ_n^2 replaced with $\ell_n^{2(d-1)}$ in (2.73) (this modification originates from the fact that a similar change takes place in (2.13) and (2.65)).

However, one cannot initiate the induction scheme one obtains in this new setup. Indeed, the corresponding probabilities $q_n(u_n)$ tend to 1 as n goes to infinity, as can be seen from the fact that $\mathbb{P}$-a.s., for large n, the box C_m at level n containing the origin meets $\mathcal{I}^{u_\infty} \subseteq \mathcal{I}^{u_n}$, and any trajectory in the interlacement entering C_m also meets $\partial_{\text{int}} \widetilde{C}_m$.

The above observation stresses the importance of being able to check the initial hypothesis of the induction and of the presence of $\mathbb{Z}^2$ in the definitions (2.3) and (2.7).

(2) Building upon Remark 2.3(1), the arguments employed in Proposition 2.4 and Theorem 2.5 also show with the appropriate adjustment of constants that given $M \geq 1$, when $L_0 \geq c(M)$, if for the choice $u_0 = c'(M)(\log L_0)^2 L_0^{-(d-2)}$ one has $c''(M)\ell_0^2 q_0(u_0) \leq L_0^{-M}$, then for all $n \geq 0$,

$$c''(M)\ell_n^2 q_n(u_\infty) \leq c''(M)\ell_n^2 q_n(u_n) \leq L_n^{-M},$$

with u_n as in (2.67) and $u_\infty = u_0 \times \prod_{n\geq 0}(1 + 1/\log L_n)$.

3 Percolation of the Vacant Set for Small u

In this section we derive the main result of this paper and show that for all $d \geq 3$, the vacant set at level u percolates in $\mathbb{Z}^2$ when u is small enough; see Theorem 3.4. The main task is to initiate the induction scheme developed in the previous section. This is carried out in Theorem 3.1 and relies on arguments that share a common flavor with some of the steps in the derivation of lower bounds for disconnection times of discrete cylinders with large boxes; cf. [1, sec. 2] and [9, sec. 5].

In this section we consider as in (2.71),

$$L_0 > c_7 \quad \text{and} \quad u_0 = \frac{4}{c_2}(\log L_0)^2 L_0^{-(d-2)}. \tag{3.1}$$

The following result is sufficient to show that we can find $L_0 > c_7$ such that (2.72) holds true. We refer to (2.8) for the notation.

THEOREM 3.1 $(d \geq 3)$

$$\lim_{L_0 \to \infty} L_0^\rho q_0(u_0) = 0 \quad \textit{for all } \rho > 0. \tag{3.2}$$

PROOF: We introduce the event

$$\begin{aligned} C_{L_0} = {} & \text{there is a self-avoiding } *\text{-path in } ([0, 2L_0) \times [0, 6L_0) \times \\ & \{0\}^{d-2}) \cap \mathcal{I}^{u_0} \text{ starting in } \{0\} \times [0, 6L_0) \times \{0\}^{d-2} \text{ and ending} \\ & \text{in } \{2L_0 - 1\} \times [0, 6L_0) \times \{0\}^{d-2}. \end{aligned} \tag{3.3}$$

Consider $m \in I_0$ (cf. (2.3)), the label of the box at level 0 containing the origin. On the event $B_m^{u_0}$ (cf. (2.7)), we can find a self-avoiding $*$-path from $\partial_{\text{int}} \widetilde{C}_m$ to ∂C_m in $(\widetilde{C}_m \setminus C_m) \cap \mathcal{I}^{u_0} \cap \mathbb{Z}^2$. One can extract from this path a self-avoiding $*$-path linking the long sides of one of the four overlapping rectangles isometric to $[0, 2L_0) \times [0, 6L_0) \times \{0\}^{d-2}$ and covering $(\widetilde{C}_m \setminus C_m) \cap \mathbb{Z}^2$; see Figure 3.1. As a consequence of (1.14) with $u = u_0$, or, alternatively, of the fact that the law of $\mathcal{I}^{u_0}$ is invariant under the discrete isometries of $\mathbb{Z}^d$, we find that

$$q_0(u_0) \leq 4\mathbb{P}[C_{L_0}]. \tag{3.4}$$

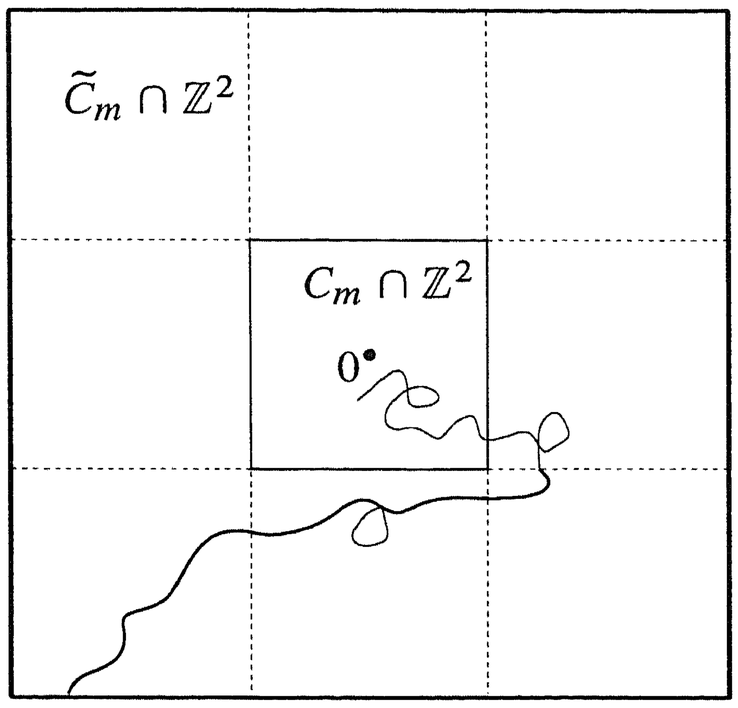

FIGURE 3.1. An illustration of the fact that when $B_m^{u_0}$ occurs, the trace of $\mathcal{I}^{u_0}$ on one of the four overlapping rectangles bordering $C_m \cap \mathbb{Z}^2$ and isometric to $[0, 2L_0) \times [0, 6L_0) \times \{0\}^{d-2}$ contains a self-avoiding $*$-path between the long sides of the rectangle.

For large L_0 we then introduce the smaller length scale

$$L = 1000[L_0 \exp\{-(\log L_0)^{1/3}\}], \tag{3.5}$$

as well as the squares in the strip $U = [0, 2L_0) \times \mathbb{Z} \times \{0\}^{d-2}$:

$$\begin{aligned} &C_{k,\ell} = kLe_1 + \ell \frac{L}{1000} e_2 + [0, L)^2 \times \{0\}^{d-2} \subseteq U \\ &\text{with } 0 \le k < K = \left[\frac{2L_0}{L}\right] \quad \text{and} \quad |\ell| < K' = \left[10^4 \frac{L_0}{L}\right]. \end{aligned} \tag{3.6}$$

For k, ℓ as above, we also consider the box

$$B_{k,\ell} = C_{k,\ell} \times [0, L)^{d-2} \ (\supseteq C_{k,\ell}), \tag{3.7}$$

and the nearly concentric subsquare of $C_{k,\ell}$,

$$C'_{k,\ell} = kLe_1 + \ell \frac{L}{1000} e_2 + \left[\frac{4L}{10}, \frac{6L}{10}\right]^2 \times \{0\}^{d-2}. \tag{3.8}$$

We denote by π_1 and π_2 the projections from $\mathbb{Z}^d$ onto $\mathbb{Z}e_1$ and $\mathbb{Z}e_2$, respectively. The next lemma is close in spirit to the geometric lemma of [1, pp. 332–334], albeit simpler due to the two-dimensional situation considered here.

LEMMA 3.2 *For large L_0 on $\mathcal{C}_{L_0}$, for each $0 \le k < K$ there exists ℓ_k with $|\ell_k| < K'$ such that*

$$|\pi_1(\mathcal{I}^{u_0} \cap C_{k,\ell_k})| \ge \frac{L}{4} \quad \text{or} \quad |\pi_2(\mathcal{I}^{u_0} \cap C_{k,\ell_k})| \ge \frac{L}{4}. \tag{3.9}$$

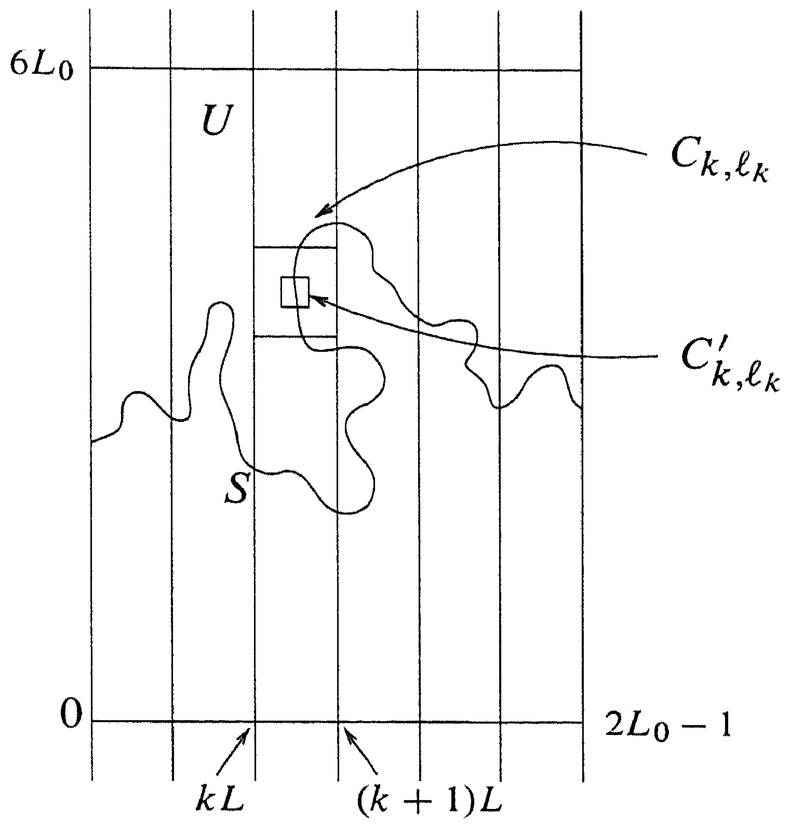

FIGURE 3.2. An illustration of the boxes C_{k,ℓ_k} and C'_{k,ℓ_k}.

PROOF: Consider $\omega \in \mathcal{C}_{L_0}$; we can find a self-avoiding $*$-path x_i, $0 \le i \le N$, as in (3.3), such that in addition $|x_i - x_j|_\infty = 1$ implies $|i-j| = 1$ whenever $0 \le i, j \le N$, and only x_0 and x_N belong to the sides of the strip U. Denote by $S = \{x_i : 0 \le i \le N\}$ the range of the path, and with $\mathcal{T}op$ the connected component in $U \setminus S$ "above the path," i.e., containing $[0, 2L_0) \times [6L_0, \infty)$. With the Jordan curve theorem for polygons (cf. [2, p. 68], one sees that $[0, 2L_0) \times (-\infty, -1] \subseteq U \setminus S$ is disjoint from $\mathcal{T}op$. When L_0 is large, it thus follows that

$$\text{for } 0 \le k < K, \quad C'_{k,\ell} \subseteq \begin{cases} \mathcal{T}op & \text{when } \ell = K' - 1, \\ \mathcal{T}op^{\mathrm{c}} & \text{when } \ell = -(K' - 1). \end{cases}$$

Tracking the relative fraction of points of $\mathcal{T}op$ in $C'_{k,\ell}$ as ℓ varies, one sees "by continuity" that for some $|\ell_k| < K'$,

$$C'_{k,\ell_k} \cap \mathcal{T}op \neq \varnothing \quad \text{and} \quad C'_{k,\ell_k} \cap (\mathcal{T}op)^{\mathrm{c}} \neq \varnothing.$$

This implies that C'_{k,ℓ_k} meets S as well. By definition of S, we can then find a $*$-path from C'_{k,ℓ_k} to $\partial_{\mathrm{int}} C_{k,\ell_k}$. Either the π_1- or π_2-projection of this path contains at least $\frac{L}{4}$ points and claim (3.9) follows; see Figure 3.2. □

As a result of the above lemma, we see that for large L_0,

$$\mathcal{C}_{L_0} \subseteq \bigcup_{(\ell_k)} \mathcal{A}_1^{(\ell_k)} \cup \mathcal{A}_2^{(\ell_k)}, \tag{3.10}$$

where (ℓ_k) runs over all maps $k \in [0, K) \to \ell_k \in (-K', K')$, with K, K' defined in (3.6), and for each (ℓ_k) we have the set

$$\mathcal{A}_i^{(\ell_k)} = \left\{ \omega \in \Omega : \left| \left\{ r \in [0, K) : |\pi_i(\mathcal{I}^{u_0} \cap C_{r,\ell_r})| \geq \frac{L}{4} \right\} \right| \geq \frac{K}{2} \right\} \quad \text{for } i = 1, 2. \tag{3.11}$$

With a rough counting argument, we thus find that for large L_0

$$\mathbb{P}[\mathcal{C}_{L_0}] \leq e^{c\frac{L_0}{L}\log\left(c\frac{L_0}{L}\right)} \sup_{(\ell_k), i=1,2} \mathbb{P}[\mathcal{A}_i^{(\ell_k)}], \tag{3.12}$$

where in the above supremum (ℓ_k) runs over the same collection as in (3.10).

We now bound $\mathbb{P}[\mathcal{A}_1^{(\ell_k)}]$ uniformly in (ℓ_k). An analogous bound for $\mathbb{P}[\mathcal{A}_2^{(\ell_k)}]$ is derived in a similar fashion. We thus consider some (ℓ_k) as above and for $0 \leq k < K$, denote with $\mathcal{S}_k$ the collection of "vertical segments" $\mathcal{I}$ of the form $\pi_1^{-1}(v) \cap C_{k,\ell_k}$ for $v \in [kL, (k+1)L) = \pi_1(C_{k,\ell_k})$. We also write $\mathcal{S} = \bigcup_{0 \leq k < K} \mathcal{S}_k$ for the collection of vertical segments in all C_{k,ℓ_k}, $0 \leq k < K$. The next step is the following (see (1.2) for the notation):

LEMMA 3.3 *For large L_0, one has*

$$\sup_{x \in \mathbb{Z}^d} E_x\left[\exp\left\{\frac{c_8}{(\log L_0/L)} \frac{\log L}{L} \sum_{I \in \mathcal{S}} \mathbb{1}_{\{H_I < \infty\}}\right\}\right] \leq 2. \tag{3.13}$$

PROOF: With Khaśminskii's lemma (cf. [4]) as well as (2.46) of [1], it suffices to prove that for large L_0,

$$\sup_{x \in \mathbb{Z}^d} E_x\left[\sum_{I \in \mathcal{S}} \mathbb{1}_{\{H_I < \infty\}}\right] \leq c \log\left(\frac{L_0}{L}\right) \frac{L}{\log L}. \tag{3.14}$$

With the help of (1.7) and classical bounds on the Green function (cf. [5, p. 31], we see that for any $0 \leq k < K$ and $x \in C_{k,\ell_k}$, one has

$$\begin{aligned} \sum_{I \in \mathcal{S}_k} P_x[H_I < \infty] &\leq \sum_{\ell=1}^{L} \frac{c}{\ell^{d-3}} && \text{if } d \geq 4, \\ &\leq \sum_{\ell=1}^{L} c \frac{1 + \log(L/\ell)}{\log L} && \text{if } d = 3. \end{aligned}$$

Hence we see that

$$\sup_{x \in C_{k,\ell_k}} E_x\left[\sum_{I \in \mathcal{S}_k} \mathbb{1}_{\{H_I < \infty\}}\right] \leq c \frac{L}{\log L} \quad \text{for } 0 \leq k < K. \tag{3.15}$$

Applying once again standard bounds on the Green function and (1.7), we find that for $x \in C \overset{\text{def}}{=} \bigcup_{0 \le k < K} C_{k,\ell_k}$, one has

$$\begin{aligned} \sum_{0 \le k < K} P_x[H_{C_{k,\ell_k}} < \infty] &\overset{(3.7)}{\le} \sum_{0 \le k < K} P_x[H_{B_{k,\ell_k}} < \infty] \\ &\le c \sum_{1 \le j \le [\frac{L_0}{L}]} \frac{L^{d-2}}{(jL)^{d-2}} \le c \log \frac{L_0}{L}. \end{aligned} \tag{3.16}$$

Thus for any $x \in \mathbb{Z}^d$ applying the strong Markov property we find that

$$\begin{aligned} &E_x\Big[\sum_{I \in \mathcal{S}} \mathbb{1}_{\{H_I < \infty\}}\Big] \\ &\quad = E_x\Big[H_C < \infty,\ E_{X_{H_C}}\Big[\sum_{I \in \mathcal{S}} \mathbb{1}_{\{H_I < \infty\}}\Big]\Big] \\ &\quad \le \sup_{z \in C} E_z\Big[\sum_{0 \le k < K} \sum_{I \in \mathcal{S}_k} \mathbb{1}_{\{H_I < \infty\}}\Big] \qquad (3.17) \\ &\quad \le \sup_{z \in C} \sum_{0 \le k < K} E_z\Big[H_{C_{k,\ell_k}} < \infty, E_{X_{H_{C_{k,\ell_k}}}}\Big[\sum_{I \in \mathcal{S}_k} \mathbb{1}_{\{H_I < \infty\}}\Big]\Big] \\ &\quad \overset{(3.15)}{\underset{(3.16)}{\le}} c \frac{L}{\log L} \log\Big(\frac{L_0}{L}\Big). \end{aligned}$$

This proves (3.14) and thus concludes the proof of the lemma. □

We now resume the task of bounding $\mathbb{P}[\mathcal{A}_1^{(\ell_k)}]$. We introduce the nonnegative measurable function on W_+:

$$\phi(w) = \frac{c_8}{\log(L_0/L)} \frac{\log L}{L} \sum_{I \in \mathcal{S}} \mathbb{1}_{\{H_I(w) < \infty\}}, \tag{3.18}$$

as well as $B = B(0, 20L_0) \supseteq C$ for large L_0 (see above (3.16) for the notation). Hence for large L_0, we find that in the notation of (1.13)

$$\begin{aligned} \mathbb{E}[\exp\{\langle \mu_{B,u_0}, \phi \rangle\}] &\overset{(1.14)}{=} \exp\{u_0 E_{e_B}[e^{\phi} - 1]\} \\ &\overset{(3.13)}{\le} \exp\{c u_0 \operatorname{cap}(B)\} \overset{(1.8)}{\underset{(3.1)}{\le}} \exp\{c(\log L_0)^2\}. \end{aligned} \tag{3.19}$$

Since on $\mathcal{A}_1^{(\ell_k)}$ one has

$$\langle \mu_{B,u_0}, \phi \rangle \ge \frac{c}{\log(L_0/L)} \frac{\log L}{L} L_0,$$

it follows that

$$\mathbb{P}[\mathcal{A}_1^{(\ell_k)}] \leq \exp\left\{-\frac{c}{\log(L_0/L)}\,\frac{\log L}{L}\,L_0 + c'(\log L_0)^2\right\}.$$

With (3.5) we also see that for large L_0,

$$c \log L_0 \leq \log L \leq c' \log L_0,$$

$$c \exp\{(\log L_0)^{1/3}\} \leq \frac{L_0}{L} \leq c' \exp\{(\log L_0)^{1/3}\}, \tag{3.20}$$

$$\log\left(\frac{L_0}{L}\right) \leq c(\log L_0)^{1/3}.$$

As a result, we deduce that for large L_0

$$\sup_{(\ell_k)} \mathbb{P}[\mathcal{A}_1^{(\ell_k)}] \leq \exp\{-c(\log L_0)^{2/3}\, e^{(\log L_0)^{1/3}}\}. \tag{3.21}$$

A similar bound holds with $\mathcal{A}_2^{(\ell_k)}$ in place of $\mathcal{A}_1^{(\ell_k)}$. Coming back to (3.12) we then see with (3.20) that for large L_0

$$\mathbb{P}[\mathcal{C}_{L_0}] \leq \exp\{-c(\log L_0)^{2/3} e^{(\log L_0)^{1/3}}\}, \tag{3.22}$$

which is more than enough to prove claim (3.2). □

We now come to the main result showing that for small u the vacant set at level u percolates in planes.

THEOREM 3.4 ($d \geq 3$) *For small $u > 0$,*

$$\mathbb{P}\text{-a.s., } \mathcal{V}^u \cap \mathbb{Z}^2 \text{ contains an infinite connected component.} \tag{3.23}$$

PROOF: The argument is in essence the same as for the proof of theorem 4.3 of [8]. With Theorem 3.1 we can pick $L_0 \geq c_7$ such that (2.72) holds and conclude with Theorem 2.5 that

$$c_1 \ell_n^2 q_n(u_\infty) \leq L_n^{-1} \quad \text{for all } n \geq 0 \text{ with } u_\infty \in (0, 1]. \tag{3.24}$$

Then for $n_0 \geq 0$ and $M = 2L_{n_0} - 1$ we can write for $u \leq u_\infty$

$$\begin{aligned}&\mathbb{P}[0 \text{ does not belong to an infinite connected component of } \mathcal{V}^u \cap \\ &\mathbb{Z}^2] \leq \mathbb{P}[\mathcal{I}^u \cap B(0, M) \cap \mathbb{Z}^2 \neq \varnothing] + \mathbb{P}[\mathcal{I}^u \cap (\mathbb{Z}^2 \setminus B(0, M)) \\ &\text{contains a } *\text{-circuit surrounding } 0].\end{aligned} \tag{3.25}$$

With (0.1) (see also (1.58) of [8]), $\mathbb{P}[x \in \mathcal{V}^u] = e^{-u/g(0)}$, with the notation as below (1.3). The above expression is thus smaller than

$$\begin{aligned}&cM^2(1 - e^{-\frac{u}{g(0)}}) + \textstyle\sum_{n \geq n_0} \mathbb{P}[\mathcal{I}^u \cap (\mathbb{Z}^2 \setminus B(0, M)) \text{ contains} \\ &\text{a } *\text{-circuit surrounding } 0, \text{ which passes through a point in} \\ &[2L_n, 2L_{n+1} - 1]e_1].\end{aligned}$$

We can cover $[2L_n, 2L_{n+1} - 1]\, e_1$ with the translates of the box C_m at level n containing the origin by the vectors $(2k + 1)\, L_n\, e_1$, $1 \leq k \leq \ell_n - 1$. A $*$-circuit

in $\mathcal{I}^u \cap (\mathbb{Z}^2 \setminus B(0, M))$ surrounding 0 and passing through $(2k+1)\, L_n\, e_1 + C_m$ necessarily meets $(2k+1) L_n\, e_1 + \partial_{\text{int}} \widetilde{C}_m$. As a result, the left-hand side of (3.25) is smaller than

$$cM^2(1 - e^{-u/g(0)}) + \sum_{n \geq n_0} \ell_n\, q_n(u) \overset{(3.24)}{\leq} c\Big(L_{n_0}^2 u + \sum_{n \geq n_0} L_n^{-1}\Big) < 1$$

if we choose n_0 large and $u \leq c(L_0, n_0)$. This yields a positive u such that

$$\mathbb{P}[0 \text{ belongs to an infinite component of } \mathcal{V}^u \cap \mathbb{Z}^2] > 0. \tag{3.26}$$

However, (2.6) of [8] shows that the law on $\{0,1\}^{\mathbb{Z}^2}$ of the indicator function of $\mathcal{V}^u \cap \mathbb{Z}^2$ is ergodic under translations. With (3.26) we see that the translation-invariant event on $\{0,1\}^{\mathbb{Z}^2}$ consisting of configurations for which the set of locations where the configuration takes the value 1 contains an infinite connected component, has full measure under this law. Claim (3.23) follows. □

Remark 3.5.

(1) As mentioned in the introduction, the above theorem combined with the results of [8] completes the proof of the nondegeneracy for all $d \geq 3$ of the critical parameter u_* of (0.3). It even proves the nondegeneracy for all $d \geq 3$ of the critical parameter attached (with a straightforward modification of (0.2)–(0.3)) to the percolation of $\mathcal{V}^u \cap \mathbb{Z}^2$. It is plausible that this critical value is strictly smaller than u_*. However, one may wonder whether it is possible to approximate u_* from below by critical values corresponding to percolation of the vacant set in thick two-dimensional slabs, as in the case of Bernoulli percolation; see [3, p. 148].

(2) With Remark 2.6(2) and a small variation in the proof of Theorem 3.1 (cf. (3.19) where the definition of u_0 involves a new constant $c'(M)$ in place of $4/c_2$), we see that for any $M \geq 1$, when $L_0 \geq c(M)$, then

$$c''(M)\ell_n^2 q_n(u_\infty) \leq c''(M)\ell_n^2 q_n(u_n) \leq L_n^{-M} \quad \text{for all } n \geq 0, \tag{3.27}$$

with u_n as in (2.67) and $u_\infty = u_0 \times \prod_{n \geq 0}(1 + 1/\log L_n)$.

As a direct consequence of this result, (see also (2.1), (2.2), (2.8)), it follows that for any $\rho > 0$, there exists $u(\rho) > 0$ such that for $u \leq u(\rho)$,

$$\lim_{L \to \infty} L^\rho \mathbb{P}[\text{there is a } *\text{-path from 0 to } S(0, L) \text{ in } \mathcal{I}^u \cap \mathbb{Z}^2] = 0. \tag{3.28}$$

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